Down-Sampler: Frequency-domain characterization

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n]$$

$$y[n] = x[Mn]$$

Take ZT on both sides

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn]\, z^{-n}$$ } This cannot be expressed in terms of $X(z)$.

To get around this problem, define $\underbrace{x_{int}[n]}_{\text{intermediate function}}$ whose sample values are the same as that of $x[n]$ at the values of n that are multiples of M and are zeros at other values of n:

$$x_{int}[n] = \begin{cases} x[n] & ; n = 0, \pm M, \pm 2M, \dots \\ 0 & ; \text{otherwise} \end{cases}$$

Therefore,

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn]\, z^{-n} = \sum_{n=-\infty}^{\infty} x_{int}[Mn]\, z^{-n}$$

By replacing Mn with l, we get

$$Y(z) = \sum_{l=-\infty}^{\infty} x_{int}[l]\, z^{-l/M} = \sum_{l=-\infty}^{\infty} x_{int}[l] \left(z^{1/M}\right)^{-l}$$

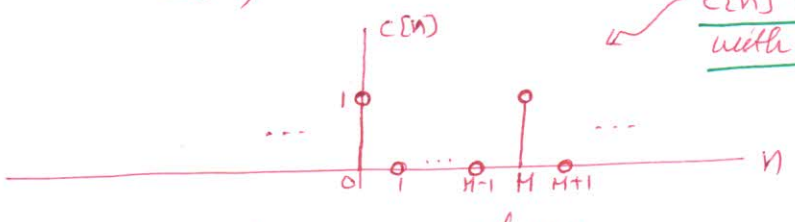
$$\Rightarrow \boxed{Y(z) = X_{int}\left(z^{1/M}\right)} \quad \text{--- (1)}$$

Now, $x_{int}[n]$ can be represented in terms of $x[n]$ by the relation

$$x_{int}[n] = c[n]\, x[n] \quad \text{--- (2)}$$

where $c[n] = 1 \;;\; n = 0, \pm M, \pm 2M, \dots$

$= 0 \;;\;$ otherwise



c[n] is periodic with period M.

By using DFS, $c[n]$ can be expressed as

$$c[n] = \sum_{k=0}^{M-1} C[k]\, e^{+jk\frac{2\pi}{M}n}$$

where $$C[k] = \frac{1}{M} \sum_{n=0}^{M-1} c[n]\, e^{-jk\frac{2\pi}{M}n} = \frac{1}{M}$$

$$\therefore\; c[n] = \sum_{k=0}^{M-1} \frac{1}{M} e^{+jk\frac{2\pi}{M}n} = \frac{1}{M} \sum_{k=0}^{M-1} e^{+jk\frac{2\pi}{M}n} \quad \text{--- (3)}$$